

Super edge-magic deficiency of join-product graphs

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Abstract

A graph G is called *super edge-magic* if there exists a bijective function f from $V(G) \cup E(G)$ to $\{1, 2, \dots, |V(G) \cup E(G)|\}$ such that $f(V(G)) = \{1, 2, \dots, |V(G)|\}$ and $f(x) + f(xy) + f(y)$ is a constant k for every edge xy of G . Furthermore, the *super edge-magic deficiency* of a graph G is either the minimum nonnegative integer n such that $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer.

Join product of two graphs is their graph union with additional edges that connect all vertices of the first graph to each vertex of the second graph. In this paper, we study the super edge-magic deficiencies of a wheel minus an edge and join products of a path, a star, and a cycle, respectively, with isolated vertices. In general, we show that the join product of a super edge-magic graph with isolated vertices has finite super edge-magic deficiency.

Keywords super edge-magic graph, super edge-magic deficiency

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1 Introduction

All graphs that we consider in this paper are finite and simple. For most graph theory notions, we refer the reader to Chartrand and Lesniak's [3]. However, to make this paper reasonably self-contained, we mention that for a graph G , we denote the vertex and edge sets of graph G by $V(G)$ and $E(G)$, respectively, and $p = |V(G)|$ and $q = |E(G)|$.

An *edge-magic labeling* of a graph G is a bijective function f from $V(G) \cup E(G)$ to $\{1, 2, \dots, p+q\}$ such that $f(x) + f(xy) + f(y)$ is a constant k , called a *magic constant* of f , for any edge xy of G . An edge-magic labeling f is called a *super edge-magic labeling* if $f(V(G)) = \{1, 2, \dots, p\}$. A graph G is called *edge-magic* (*super edge-magic*) if there exists an edge-magic (super edge-magic, respectively) labeling of G . The concept of edge-magic labeling was first introduced by Kotzig and Rosa [10] and the super edge-magic labeling was introduced by Enomoto, Lladó, Nakamigawa and Ringel [4]. We mention that an equivalent concept to the one of super edge-magic graphs had already appeared in the literature under the name of strongly indexable graphs [1]. Although the definitions of super edge-magic graphs and strongly indexable graphs were introduced from different points of view, they turn out to be equivalent.

In [10], Kotzig and Rosa proved that for every graph G there exists an edge-magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n . This fact motivated them to define the concept of edge-magic deficiency of a graph. The *edge-magic deficiency* of a graph G , $\mu(G)$, is defined as the minimum nonnegative integer n such that $G \cup nK_1$ is edge-magic. They also proved that every graph has finite edge-magic deficiency. Motivated by Kotzig and Rosa's concept, Figueroa-Centeno *et al.* [6] defined a similar concept for super edge-magic labelings. The *super edge-magic deficiency* of a graph G , $\mu_s(G)$, is either the minimum nonnegative integer n such that $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer. As a direct consequence of the above two definitions, the inequality $\mu(G) \leq \mu_s(G)$ holds for every graph G .

Some authors have studied the super edge-magic deficiency of some classes of graphs. Figueroa-Centeno *et al.* in two separate papers [6, 7] investigated super edge-magic deficiencies of complete graphs, complete bipartite graphs $K_{2,m}$, some classes of forests with two components, 1-regular graphs, and 2-regular graphs. Ngurah *et al.* [11, 12] studied the super edge-magic deficiency of some classes of chain graphs, wheels, fans, double fans, and disjoint union of particular type of complete bipartite graphs. Recently, Ahmad and Muntaner-Battle [2] studied the super edge-

magic deficiency of several classes of unicyclic graphs. The authors refer the reader to the survey paper by Gallian [9] for some of the latest developments in these and other types of graph labelings.

In this paper, we study the super edge-magic deficiencies of a wheel minus an edge and join products of a path, a star, and a cycle, respectively, with isolated vertices. In proving the main results, the following two lemmas will be used frequently. The first lemma characterizes super edge-magic graphs and the second gives necessary conditions for the existence of super edge-magic graphs.

Lemma 1 [5] *A graph G with p vertices and q edges is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \dots, p\}$ such that the set $S = \{f(x) + f(y) : xy \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic total labeling of G with the magic constant $k = p + q + s$, where $s = \min(S)$.*

Lemma 2 [4] *If a graph G with p vertices and q edges is super edge-magic, then $q \leq 2p - 3$.*

2 Super edge-magic deficiency of a wheel minus an edge

In this section, we consider the super edge-magic deficiency of $W_n \cong C_n + K_1$, $n \geq 3$, minus an edge. We shall denote vertex-set of W_n , $V(W_n) = \{c\} \cup \{x_1, x_2, x_3, \dots, x_n\}$, and edge-set $E(W_n) = \{cx_i : 1 \leq i \leq n\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$. We shall call an edge $x_i x_{i+1}$ as a *rim* and an edge cx_i as a *spoke*. Let us consider the graph $H_n \cong W_n - \{e\}$ with order $n+1$ and size $2n-1$. It is interesting to mention that $H_n \cong W_n - \{e\}$ is a graph attaining $|E(H_n)| = 2|V(H_n)| - 3$, which is the upper bound of condition in Lemma 2. If the edge e is a rim of W_n , then H_n is a fan F_n whose super edge-magic deficiency has been studied by Ngurah *et al.* [11]. They determined the super edge-magic deficiency of F_n for small values of n and provided upper and lower bounds for general n . Here, we consider the super edge-magic deficiency of $H_n \cong W_n - \{e\}$, where e is a spoke of W_n . We shall use the following notations for vertex and edge sets: $V(H_n) = \{c\} \cup \{x_1, x_2, x_3, \dots, x_n\}$, and edge-set $E(H_n) = \{cx_i : 2 \leq i \leq n\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$.

Our first result gives the only two super edge-magic labelings for H_n .

Theorem 1 *Let $n \geq 3$ be an integer. The graph $H_n \cong W_n - \{e\}$ is super edge-magic if and only if $n \leq 4$.*

Proof First, we show that H_n is super edge-magic for $n \leq 4$. Label the vertices $(c; x_1, x_2, x_3)$ and $(c; x_1, x_2, x_3, x_4)$ with $(1; 4, 3, 2)$ and $(2; 3, 1, 4, 5)$, respectively. This vertex labeling extends to a super edge-magic labeling of H_3 and H_4 , respectively.

For the necessity, assume that H_n is super edge-magic with a super edge-magic labeling f for every integer $n \geq 5$. By Lemma 1, $S = \{f(u) + f(v) : uv \in E(H_n)\}$ is a set of $|E(H_n)| = 2|V(H_n)| - 3$ consecutive integers. Thus $S = \{3, 4, 5, \dots, 2n, 2n + 1\}$. We shall consider two cases.

Case 1: $n = 5, 6$. For $n = 5$, The sum of all elements of S is 63. This sum contains two times of label x_1 , three times each label of x_i , $2 \leq i \leq 5$ and four times the label of c . Thus, we have

$$\sum_{i=2}^5 f(x_i) + 2f(c) = 21.$$

It is a routine procedure to verify that this equation has no solution. Hence, H_5 is not super edge-magic. With a similar argument, for $n = 6$, we have

$$\sum_{i=2}^6 f(x_i) + 3f(c) = 32.$$

The possible solutions for this equation are $f(c) = 3, f(x_1) = 2, f(x_i) \in \{1, 4, 5, 6, 7\}, 2 \leq i \leq 6$, and $f(c) = 5, f(x_1) = 6, f(x_i) \in \{1, 2, 3, 4, 7\}, 2 \leq i \leq 6$. It can be checked that these solutions do not lead to a super edge-magic labeling of H_6 . Hence, H_6 is not a super edge-magic graph.

Case 2: $n \geq 7$. Observe that both 3 and 4 can be expressed uniquely as sums of two distinct element from the set $\{1, 2, 3, \dots, n + 1\}$, namely $3 = 1 + 2$ and $4 = 1 + 3$. On the other hand, 5 can be expressed as sums of distinct elements of $\{1, 2, \dots, n + 1\}$ in exactly two ways, namely $5 = 2 + 3 = 1 + 4$. Then, the vertices of labels 1, 2 and 3 must form a triangle or the vertex of label 1 is adjacent to the vertices of labels 2, 3 and 4, respectively. With a similar argument, the vertices of labels $n - 1$, n and $n + 1$ must form a triangle or the vertex of label $n + 1$ is adjacent to the vertices of labels $n, n - 1$ and $n - 2$, respectively. By combining these facts, we obtain either $2K_3$, $K_3 \cup K_{1,3}$ or $2K_{1,3}$ as a subgraph of H_n , a contradiction. This completes the proof. \square

Based on the results of Theorem 1, the super edge-magic deficiency of H_n is 0 for $n = 3$ and 4, and at least 1 for $n \geq 5$. For $n = 5, 6, 7$, we

could prove that $\mu_s(H_n) = 1$ by labeling the vertices $(c; x_1, x_2, x_3, x_4, x_5)$, $(c; x_1, x_2, x_3, x_4, x_5, x_6)$, and $(c; x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ with $(1; 7, 5, 3, 6, 4)$, $(2; 3, 1, 4, 8, 5, 6)$, and $(2; 3, 1, 4, 8, 5, 9, 6)$, respectively.

For $n \geq 8$ we shall determine an upper bound for the super edge-magic deficiency of H_n where $n \not\equiv 2 \pmod{4}$ as stated in the following theorem.

Theorem 2 *For any integer $n \geq 8$, $n \equiv 0, 1, 3 \pmod{4}$, the super edge-magic deficiency of H_n are given by*

$$\mu_s(H_n) \leq \begin{cases} \frac{1}{2}(n-3), & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}, \\ \frac{n}{2}, & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Proof We consider the following two cases.

Case 1: $n \equiv 1$ or $3 \pmod{4}$. Define a vertex labeling as follow.

$$f(c) = \frac{1}{2}(3n-1).$$

$$f(x_i) = \begin{cases} \frac{1}{2}(i+1), & \text{if } i = 1, 3, 5, \dots, n-1, \\ \lceil \frac{n}{2} \rceil + \frac{i}{2}, & \text{if } i = 2, 4, 6, \dots, n-2. \end{cases}$$

Case 2: $n \equiv 0 \pmod{4}$. We redefine the edge-set of H_n as $E(H_n) = \{cx_i : 1 \leq i \leq n, i \neq \frac{n}{2}\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$. Now we are ready to define a vertex labeling f .

$$f(c) = \frac{1}{2}(3n+2).$$

$$f(x_i) = \begin{cases} \frac{1}{2}(i+1), & \text{if } i = 1, 3, 5, \dots, n-1, \\ \frac{1}{2}(n+i), & \text{if } i = 2, 4, 6, \dots, \frac{1}{2}(n-4), \\ \frac{5}{4}n, & \text{if } i = \frac{n}{2}, \\ \frac{1}{2}(n+i-2), & \text{if } i = \frac{n}{2} + 2, \frac{n}{2} + 4, \dots, n. \end{cases}$$

For both cases, it is easy to verify that f extends to a super edge-magic labeling of H_n . \square

We have tried to find an upper bound of the super edge-magic deficiency of H_n for $n \equiv 2 \pmod{4}$, but without success. And thus we propose the following problems.

Open problem 1 *For $n \equiv 2 \pmod{4}$, find an upper bound of the super edge-magic deficiency of H_n . Further, find the super edge-magic deficiency of H_n for all n .*

3 Super edge-magic deficiency of join-product graphs

In this section, we consider super edge-magic deficiency of three classes of graphs. These graphs are obtained from join products of a path P_n , a star $K_{1,n}$, and a cycle C_n , respectively, with m isolated vertices ($\overline{K_m}$).

First, we consider the super edge-magic deficiency of $P_n + \overline{K_m}$. We denote the vertex and edge sets of $P_n + \overline{K_m}$ as

$$V(P_n + \overline{K_m}) = \{u_i : 1 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq m\}$$

and

$$E(P_n + \overline{K_m}) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_j : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

It is clear that $P_n + \overline{K_m}$ is a graph of order and size $n+m$ and $n(m+1)-1$, respectively.

If $m = 1$, then $P_n + \overline{K_1}$ is a fan F_n . As we mention in the first section, the super edge-magic deficiency of F_n have been studied in [11]. Furthermore, Ngurah *et al.* [12] studied the super edge-magic deficiency of $P_n + \overline{K_2}$ and proved that $\mu_s(P_n + \overline{K_2}) = \frac{1}{2}(n-2)$ for all even $n \geq 2$, and conjectured that $\mu_s(P_n + \overline{K_2}) = \frac{1}{2}(n-1)$ for all odd $n \geq 3$. In this section, we study the super edge-magic deficiency of $P_n + \overline{K_m}$ for $m \geq 3$. The next result provides sufficient and necessary conditions for $P_n + \overline{K_m}$ to be super edge-magic.

Lemma 3 *Let $n \geq 1$ and $m \geq 3$ be integers. Then the graph $P_n + \overline{K_m}$ is super edge-magic if and only if $n \in \{1, 2\}$.*

Proof First, we show that $P_n + \overline{K_m}$ is super edge-magic for $n = 1, 2$. It is known that $P_1 + \overline{K_m} \cong K_{1,m}$ is super edge-magic. For $n = 2$ label the vertices $\{u_1, u_2\}$ and $\{v_1, v_2, v_3, \dots, v_m\}$ with $\{1, m+2\}$ and $\{2, 3, \dots, m+1\}$, respectively. Then by Lemma 1, this vertex labeling extends to a super edge-magic labeling of $P_2 + \overline{K_m}$ with the magic constant $3m+6$. For the sufficiency, let $P_n + \overline{K_m}$ be a super edge-magic graph. By Lemma 2, we have $n(m+1)-1 \leq 2(n+m)-3$ and the desired result. \square

Based on Lemma 3, $\mu_s(P_n + \overline{K_m}) = 0$ for $n \leq 2$ and $\mu_s(P_n + \overline{K_m}) \geq 1$ for $n \geq 3$. Since there is no super edge-magic labeling of $P_n + \overline{K_m}$ for almost all values of n , we thus try to find its super edge-magic deficiency. The following theorem gives upper and lower bounds of the deficiency.

Theorem 3 For any integers $n, m \geq 3$, the super edge-magic deficiency of $P_n + \overline{K_m}$ satisfies $\lceil \frac{1}{2}(n-2)(m-1) \rceil \leq \mu_s(P_n + \overline{K_m}) \leq (n-1)(m-1) - 1$.

Proof To prove the upper bound, we define a vertex labeling f as follow.

$$f(u_i) = \begin{cases} \lfloor \frac{1}{2}(n+2) \rfloor + \frac{1}{2}(i-1), & \text{for odd } i, \\ n + \frac{1}{2}i, & \text{for even } i, \end{cases}$$

and

$$f(\{v_1, v_2, v_3, \dots, v_m\}) = \{1, 2n, 3n, 4n, \dots, mn\}.$$

We can see that these vertex-labels are non-repeated and constitute a set $\{f(x) + f(y) | xy \in E(P_n + \overline{K_m})\}$ of $n(m+1) - 1$ consecutive integers. However, the largest vertex label used is mn and there exist $mn - (n+m) = (n-1)(m-1) - 1$ labels that are not utilized. So, for each number between 1 and mn that has not been used as a label, we introduce a new vertex labeled with that number; and this gives $(n-1)(m-1) - 1$ isolated vertices. By Lemma 1, this yields a super edge-magic labeling of $P_n + \overline{K_m} \cup [(n-1)(m-1) - 1]K_1$ with magic constant $2mn + \lfloor \frac{1}{2}(3n+2) \rfloor$. Hence,

$$\mu_s(P_n + \overline{K_m}) \leq (n-1)(m-1) - 1.$$

For a lower bound, by Lemma 2, it is easy to verify that

$$\mu_s(P_n + \overline{K_m}) \geq \lceil \frac{1}{2}(n-2)(m-1) \rceil. \quad \square$$

Notice that, the lower bound presented in Theorem 3 is tight. We found that the super edge-magic deficiency of $P_4 + \overline{K_m}$ is equal to its lower bound by labeling the vertices (u_1, u_2, u_3, u_4) and $\{v_1, v_2, v_3, \dots, v_m\}$ with $(1, 2, 2m+2, 2m+3)$ and $\{3, 5, 7, \dots, 2m-1, 2m+1\}$, respectively. This vertex-labels extend to a super edge-magic labeling of $P_4 + \overline{K_m}$ with the magic constant $6m+9$. The largest vertex label used is $2m+3$. So, $\mu_s(P_4 + \overline{K_m}) \leq 2m+3 - (m+4) = m-1$. From this fact and Theorem 3, $\mu_s(P_4 + \overline{K_m}) = m-1$. Additionally, we found that $\mu_s(P_6 + \overline{K_m}) = 2(m-1)$ by labeling the vertices $(u_1, u_2, u_3, u_4, u_5, u_6)$ and $\{v_1, v_2, v_3, \dots, v_m\}$ with $(2, 1, 3, 3m+2, 3m+4, 3m+3)$ and $\{4, 7, 10, \dots, 2m-5, 2m-2, 3m+1\}$, respectively.

Referring to the afore-mentioned results, we propose the following problems.

Open problem 2 Find a better upper bound of the super edge-magic deficiency of $P_n + \overline{K_m}$. Further, find the super edge-magic deficiency of $P_n + \overline{K_m}$ for $n \neq 4, 6$.

Let us now determine the super edge-magic deficiency of $K_{1,n} + \overline{K_m}$. Let $K_{1,n} + \overline{K_m}$ be a graph having

$$V(K_{1,n} + \overline{K_m}) = \{c\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq m\},$$

and

$$E(K_{1,n} + \overline{K_m}) = \{cx_i : 1 \leq i \leq n\} \cup \{x_i y_j, cy_j : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Thus $K_{1,n} + \overline{K_m}$ is a graph of order $n + m + 1$ and size $(n + 1)(m + 1) - 1$. Notice that if $n = 1$, then $K_{1,1} + \overline{K_m} \cong P_2 + \overline{K_m}$ which is super edge-magic (see Theorem 3). Hence, we assume that $n \geq 2$.

Lemma 4 *Let $n \geq 2$ and $m \geq 1$ be integers. Then, $K_{1,n} + \overline{K_m}$ is super edge-magic if and only if $m = 1$.*

Proof By Lemma 2, it is easy to check that if $K_{1,n} + \overline{K_m}$ is super edge-magic then $m \leq 1$. Since m is a positive integer, so $m = 1$. For the sufficiency, label the vertices $\{c\}$, $\{x_1, x_2, x_3, \dots, x_n\}$, and $\{y_1\}$ with $\{n + 1\}$, $\{1, 2, 3, \dots, n\}$, and $\{n + 2\}$, respectively. This vertex labeling extends to a super edge-magic labeling of $K_{1,n} + \overline{K_m}$ with magic constant $3n + 6$. \square

Since $K_{1,n} + \overline{K_m}$ is not super edge-magic for almost all values of m , we thus try to find its super edge-magic deficiency. The following result gives upper and lower bounds of the deficiency.

Theorem 4 *For any integers $n, m \geq 2$, the super edge-magic deficiency of $K_{1,n} + \overline{K_m}$ satisfies $\lceil \frac{1}{2}(n - 1)(m - 1) \rceil \leq \mu_s(K_{1,n} + \overline{K_m}) \leq n(m - 1) - 1$.*

Proof Similar with the proof of Theorem 3, we could obtain that $\mu_s(K_{1,n} + \overline{K_m}) \geq \lceil \frac{1}{2}(n - 1)(m - 1) \rceil$. To show the upper bound, label the vertices $\{c\}$, $\{x_1, x_2, x_3, \dots, x_n\}$, and $\{y_1, y_2, y_3, \dots, y_m\}$ with $\{n + 2\}$, $\{2, 3, 4, \dots, n + 1\}$, and $\{1, 2(n + 1), 3(n + 1), \dots, m(n + 1)\}$, respectively. This vertex labeling extends to a super edge-magic labeling of $K_{1,n} + \overline{K_m}$ with magic constant $(n + 1)(m + 1) + 1$ and the largest vertex label $m(n + 1)$. \square

Open problem 3 *For integers $n, m \geq 2$, find better upper and lower bounds of the super edge-magic deficiency of $K_{1,n} + \overline{K_m}$. Further, find the super edge-magic deficiency of $K_{1,n} + \overline{K_m}$ for a fixed value of n or m .*

Finally, we consider the super edge-magic deficiency of $C_n + \overline{K_m}$. Notice that this graph is not super edge-magic for all integers $n \geq 3$ and $m \geq 1$.

For $m = 1$, the graph $C_n + \overline{K_1}$ is a wheel W_n . Ngurah *et al.* [11] studied the super edge-magic deficiency of W_n and they determined the super edge-magic deficiency of W_n for some values of n and gave a lower bound for general values of n . Additionally, they also provided an upper bound for the super edge-magic deficiency of W_n for odd n . Now, we study the super edge-magic deficiency of $C_n + \overline{K_m}$ for $n \geq 3$ and $m \geq 2$. Our first result gives a lower bound of the super edge-magic deficiency of $C_n + \overline{K_m}$.

Lemma 5 *For any integers $n \geq 3$ and $m \geq 2$, $\mu_s(C_n + \overline{K_m}) \geq \lfloor \frac{1}{2}(m+1)n \rfloor - (n+m) + 2$.*

Proof It is easy to verify that $C_n + \overline{K_m} \cup tK_1$, where $t = \lfloor \frac{1}{2}(m+1)n \rfloor - (n+m) + 1$, is not a super edge-magic graph. Hence, $\mu_s(C_n + \overline{K_m}) \geq \lfloor \frac{1}{2}(m+1)n \rfloor - (n+m) + 2$. \square

Theorem 5 *Let $n \geq 3$ be an odd integer. Then $\mu_s(C_n + \overline{K_m}) \leq mn - (n+m) + 1$ for every integer $m \geq 2$.*

Proof Let $C_n + \overline{K_m}$ be a graph with

$$V(C_n + \overline{K_m}) = \{u_i : 1 \leq i \leq n\} \cup \{v_j : 1 \leq j \leq m\}$$

and

$$E(C_n + \overline{K_m}) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_i v_j : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Next, define a vertex labeling f as follow.

$$f(u_i) = \begin{cases} \frac{1}{2}(n+2+i), & \text{if } i = 1, 3, 5, \dots, n, \\ \frac{1}{2}(2n+2+i), & \text{if } i = 2, 4, 6, \dots, n-1, \end{cases}$$

$$f(\{v_1, v_2, v_3, \dots, v_m\}) = \{1, 2n+1, 3n+1, 4n+1, \dots, mn+1\}.$$

It is a routine procedure to check that f can be extended to a super edge-magic labeling of $C_n + \overline{K_m} \cup tK_1$, where $t = mn - (n+m) + 1$. Thus, we have the desired result. \square

Some open problems related the super edge-magic deficiency of $C_n + \overline{K_m}$ are presented bellow.

Open problem 4 *For even $n \geq 4$ and every $m \geq 2$, find an upper bound for the super edge-magic deficiency of $C_n + \overline{K_m}$. Further, find a better upper bound of the super edge-magic deficiency of $C_n + \overline{K_m}$ for odd n and every $m \geq 2$.*

Our results showed the finiteness of super edge-magic deficiencies of join product of a path, a star, and a cycle with isolated vertices. Recall that all paths, stars, and cycles of odd order are super edge-magic. In the next theorem, we managed to generalize similar result for any super edge-magic graph.

Theorem 6 *Let G be a super edge-magic graph with a super edge-magic labeling f . For any integer $m \geq 1$, $\mu_s(G + \overline{K_m}) \leq s + (m - 2)|V(G)| - m$, where $s = \max\{f(u) + f(v) : uv \in E(G)\}$.*

Proof First, define $H \cong G + \overline{K_m}$ as a graph with $V(H) = V(G) \cup \{y_1, y_2, y_3, \dots, y_m\}$ and $E(H) = E(G) \cup \{xy_i : x \in V(G), 1 \leq i \leq m\}$. Next, define a vertex labeling g as follows.

$$g(x) = f(x), \text{ if } x \in V(G),$$

and

$$g(\{y_1, y_2, y_3, \dots, y_m\}) = \{s, s + |V(G)|, s + 2|V(G)|, \dots, s + (m - 1)|V(G)|\}.$$

It is easy to verify that g extends to a super edge-magic labeling of $H \cup [s + (m - 2)|V(G)| - m]K_1$. Hence, $\mu_s(H) \leq s + (m - 2)|V(G)| - m$. \square

To conclude, we would like to ask an interesting general question regarding the super edge-magic deficiency of join-product graphs.

Open problem 5 *If G is an arbitrary graph, determine the super edge-magic deficiency of the join-product of G with m isolated vertices, $\mu_s(G + \overline{K_m})$.*

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